Closing Wed: HW_9A, 9B (9.3, 9.4)

9.4 Diff. Eq. Apps *(continued)*

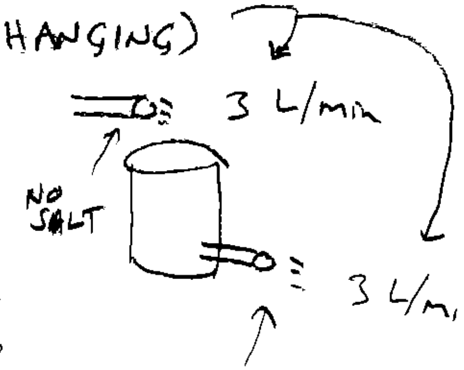
Entry Task: Assume a 12 Liter vat contains 7 kg of salt initially. A pipe pumps in *pure water* at 3 L/min. The vat is well mixed. The mixture drains at 3 L/min.

Let $y(t)$ = "kg of salt in vat at time t".

Identify and label the following:

1. Volume of the vat (Is it changing?)
2. Amount of salt per min entering.
3. Amount of salt per min exiting.
4. Initial amount of salt.

1. $V = 12$ (NOT CHANGING)



2. $\text{RATE IN} = 0 \ \dfrac{\text{kg of salt}}{\text{min}}$

3. $\text{RATE OUT} = \dfrac{y}{12} \dfrac{\text{kg}}{\text{L}} \cdot 3 \dfrac{\text{L}}{\text{min}} = \dfrac{1}{4} y$

current concentration $= \dfrac{\text{current salt}}{12 \ \text{L}}$

4. $y(0) = 7$

$$\boxed{\frac{dy}{dt} = 0 - \frac{1}{4} y \ , \ y(0) = 7}$$

$$\int \frac{1}{y}\, dy = \int -\frac{1}{4}\, dt$$

$$\ln|y| = -\frac{1}{4} t + C_1$$

$$|y| = e^{\left(-\frac{1}{4}t + C_1\right)}$$

$$y(t) = \pm e^{C_1} e^{-1/4 t} \qquad C = \pm e^{C_1}$$

$$\boxed{y(t) = C e^{-1/4 t}}$$

$$y(0) = 7 \Rightarrow C = 7$$

$$\boxed{y(t) = 7 e^{-1/4 t}}$$

Example:

Assume a 12 Liter vat contains 7 kg of salt initially. A pipe pumps in salt water (brine) at 3 L/min with a concentration of 2 kg/L of salt.

The vat is well mixed.

The mixture leaves the vat at 3 L/min.

Let $y(t)$ = the amount of salt in the vat at time t.

(a) Find $y(t)$.

(b) Find the limit of $y(t)$ as $n \to \infty$.

$\boxed{\text{RATE IN}}$ $\quad 2\,\frac{kg}{L} \; 3\,\frac{L}{min} = 6\,\frac{kg\text{ of salt}}{min}$

$\boxed{\text{RATE OUT}}$ $\quad \frac{y}{12}\,\frac{kg}{L} \; 3\,\frac{L}{m} = \frac{3y}{12} = \frac{y}{4}\,\frac{kg}{m}$

$$\frac{dy}{dt} = 6 - \frac{1}{4}y, \quad y(0) = 7$$

$$\int \frac{1}{6 - \frac{1}{4}y}\,dy = \int dt$$

$$-4\ln\left|6 - \tfrac{1}{4}y\right| = t + C_1 \qquad C_2 = -\tfrac{1}{4}C_1$$

$$\ln\left|6 - \tfrac{1}{4}y\right| = -\tfrac{1}{4}t + C_2$$

$$\left|6 - \tfrac{1}{4}y\right| = e^{C_2}e^{-\frac{1}{4}t}$$

$$6 - \tfrac{1}{4}y = \pm e^{C_2}e^{-\frac{1}{4}t} \qquad C_3 = \pm e^{C_2}$$

$$6 = C_3 e^{-\frac{1}{4}t} + \tfrac{1}{4}y$$

$$\tfrac{1}{4}y = 6 - C_3 e^{-\frac{1}{4}t} \qquad C = -4C_3$$

$$\boxed{y = 24 + Ce^{-\frac{1}{4}t}}$$

$$y(0) = 7 \Rightarrow 7 = 24 + C \Rightarrow C = +17$$

$$\boxed{y(t) = 24 - 17e^{-\frac{1}{4}t}}$$

$$\boxed{\lim_{t\to\infty} y(t) = 24}$$

Here is what these problems typically look like:

V	= volume of vat	(liters)
t	= time	(min)
$y(t)$	= amount in vat	(kg)
$\frac{dy}{dt}$	= rate	(kg/min)

Thus,

$$\frac{dy}{dt} = \text{Rate In} - \text{Rate out}$$

$$= \left(?\frac{kg}{L}\right)\left(?\frac{L}{min}\right) - \left(\frac{y}{V}\ \frac{kg}{L}\right)\left(?\frac{L}{min}\right)$$

$$y(0) = ?\ \text{kg}$$

Example:

Assume a 100 Liter vat contains 5kg of salt initially. Two pipes (A & B) pump in salt water (brine).

Pipe A: Enters at 3L/min with a concentration of 4kg/L of salt.

Pipe B: Enters at 5L/min with a concentration of 2kg/L of salt.

The vat is well mixed.

The mixture leaves the vat at 8L/min.

Let $y(t)$ = the amount of salt in the vat at time t.

(a) Find $y(t)$.

(b) Find the limit of $y(t)$ as $n \to \infty$.

$\boxed{\text{RATE IN}}$ $\underbrace{\frac{4\,kg}{L}\,\frac{3\,L}{min}}_{12} + \underbrace{\frac{2\,kg}{L}\,\frac{5\,L}{min}}_{10} = 22\ \frac{kg\ of\ salt}{min}$

$\boxed{\text{RATE OUT}}$ $\frac{y}{100}\frac{kg}{L}\,8\frac{L}{min} = \frac{8y}{100} = \frac{2y}{25}\ \frac{kg}{min}$

$$\frac{dy}{dt} = 22 - \frac{2y}{25}, \quad y(0) = 5$$

$$\int \frac{1}{22 - \frac{2y}{25}}\,dy = \int dt$$

$$-\frac{25}{2}\ln\left|22 - \frac{2y}{25}\right| = t + C_1 \qquad C_2 = -\frac{2}{25}C_1$$

$$\ln\left|22 - \frac{2y}{25}\right| = -\frac{2}{25}t + C_2$$

$$22 - \frac{2y}{25} = \pm e^{C_2} e^{-\frac{2}{25}t} \qquad C_3 = \pm e^{C_2}$$

$$22 - \frac{2y}{25} = C_3 e^{-\frac{2}{25}t}$$

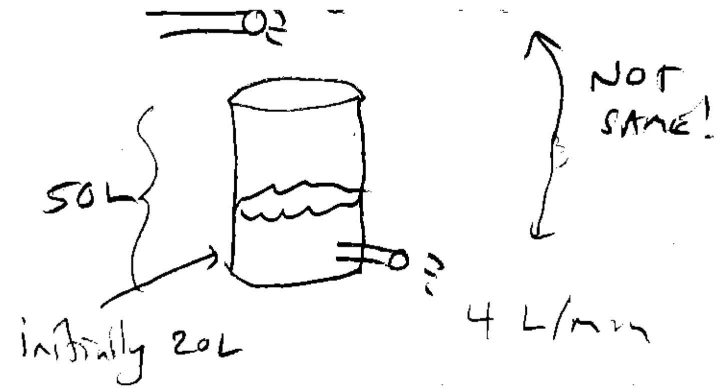
$$\frac{2y}{25} = 22 - C_3 e^{-\frac{2}{25}t} \qquad C = -\frac{25}{2}C_3$$

$$\boxed{y = 275 + Ce^{-\frac{2}{25}t}}$$

$y(0) = 5 \Rightarrow C = -270$

$$\boxed{\lim_{t\to\infty} y(t) = 275\ kg}$$

Example: Assume a 50 Liter container currently has 20 Liters of water with 24 kg of dissolved salt.
A pipe pumps in *pure water* at 6 L/min.
The vat is well mixed.
The mixture drains at 4 L/min.
Let $y(t)$ = "kg of salt in vat at time t".
What is different about this problem?



$$\text{VOLUME} = \boxed{V(t) = 20 + 2t}$$

why? $6 - 4 = 2\ \frac{L}{min}$ added

$$\text{RATE IN} = 0\ \frac{kg}{L} \cdot 6\ \frac{L}{min} = 0\ \frac{kg}{min}$$

$$\text{RATE OUT} = \frac{y}{20+2t}\ \frac{kg}{L} \cdot 4\ \frac{L}{min} = \frac{4y}{20+2t} = \frac{2y}{10+t}\ \frac{kg}{L}$$

$$\frac{dy}{dt} = 0 - \frac{2y}{10+t}, \quad y(0) = 24$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int -\frac{2}{10+t}\,dt$$

$$\ln|y| = -2\ln|10+t| + C_1$$

$$\Rightarrow |y| = e^{(-2\ln(10+t) + C_1)}$$

$$y = \pm e^{-2\ln(10+t)} e^{C_1} \qquad C = \pm e^{C_1}$$

$$y = C\,\underbrace{e^{-2\ln(10+t)}}$$

$$y = C(10+t)^{-2}$$

$$\boxed{y = \frac{C}{(10+t)^2}}$$

$$y(0) = 24 \Rightarrow \frac{C}{100} = 24 \qquad C = 2400$$

$$y(t) = \frac{2400}{\ldots}$$

$\boxed{\text{NOTE}}$

$$-2\ln(10+t) = \ln\left((10+t)^{-2}\right)$$

$$e^{\ln((10+t)^{-2})} = (10+t)^{-2}$$

4. Air Resistance:

A skydiver steps out of a plane that is 4,000 meters high with an initial downward velocity of 0 m/s.
The skydiver has a mass of 60 kg.
(Treat downward as positive).
Let $y(t)$ = "height at time *t*"

Newton's 2nd Law says:

(mass)(acceleration) = Force

$$m\frac{d^2y}{dt^2} = \text{sum of forces on the object}$$

The force due to gravity has constant magnitude (acting downward):
$F_g = mg = 60 \cdot 9.8 = 588$ N

TERMINAL VELOCITY

$$\lim_{t\to\infty} v(t) = 49 \ \text{m/sec}$$

One model for air resistance

The force due to air resistance (*drag force*) is proportional to velocity and in the opposite direction of velocity. So

$$F_d = -k\,v \quad \text{Newtons}$$

Assume for this problem k = 12.

$$m\frac{dv}{dt} = mg - kv$$

$$\int \frac{m}{mg-kv}\,dv = \int dt$$

$$-\frac{m}{k}\ln|mg-kv| = t + C_1$$

$$\ln|mg-kv| = -\frac{k}{m}t - \frac{k}{m}C_1 \qquad C_2 = -\frac{k}{m}C_1$$

$$mg-kv = \pm e^{C_2}e^{-\frac{k}{m}t} \qquad C_3 = \pm e^{C_2}$$

$$kv = mg - C_3e^{-k/m\,t}$$

$$\boxed{v(t) = \frac{mg}{k} + Ce^{-\frac{k}{m}t}} \qquad C = -\frac{1}{k}C_3$$

$$v(t) = \frac{588}{12} + Ce^{-\frac{12}{60}t} = 49 + Ce^{-1/5\,t}$$

$$V(0) = 0 \Rightarrow C = -49$$

$$\boxed{v(t) = 49 - 49e^{-1/5\,t}}$$

Spring 2011 Final:

$v(t)$ = velocity of an object

$$F = mg - kv$$

Recall:

$$F = ma = m\frac{dv}{dt}$$

You are given m, g, and k and asked for solve for $v(t)$.

DONE!

SEE LAST PAGE!

Spring 2014 Final:

A lake has a volume of 1000 cubic meters and contains pure water. A farm near the lake begins using a pesticide. Runoff from the farmland into the lake is 10 cubic meters of water per day, with a concentration of 50 grams of pesticide per cubic meter of water. The lake drains to the ocean at a rate of 10 cubic meters per day.

Find the formula *p(t)* for the amount of pesticide in the late at time *t* days.

$\boxed{\text{RATE IN}}$ $50\,\frac{g}{m^3}\cdot 10\,\frac{m^3}{day} = 500\,\frac{\text{grams}}{day}$

$\boxed{\text{RATE OUT}}$ $\frac{p}{1000}\,\frac{g}{m^3}\cdot 10\,\frac{m^3}{day} = \frac{1}{100}p\,\frac{\text{grams}}{day}$

$$\boxed{\frac{dp}{dt} = 500 - \frac{1}{100}p, \quad p(0) = 0}$$

$$\int \frac{1}{500-\frac{1}{100}p}\,dp = \int dt$$

$$-100\ln\left|500-\frac{1}{100}p\right| = t + C_1$$

$$\ln\left|500-\frac{1}{100}p\right| = -\frac{1}{100}t + C_2 \qquad C_2 = -\frac{1}{100}C_1$$

$$500-\frac{1}{100}p = \pm e^{C_2}e^{-\frac{1}{100}t} \qquad C_3 = \pm e^{C_2}$$

$$\frac{1}{100}p = 500 - C_3 e^{-\frac{1}{100}t}$$

$$\boxed{p = 50{,}000 + Ce^{-\frac{1}{100}t}} \qquad C = -100C_3$$

$$p(0)=0 \Rightarrow \boxed{p(t) = 50{,}000 - 50{,}000e^{-\frac{1}{100}t}}$$

Winter 2011 Final:

Your friend wins the lottery, and gives you P_0 dollars to help you pay your college expenses. The money is invested in a savings account that earns 10% annual interest, compounded continuously, and you withdraw the money continuously at an average rate of $3600 per year.

Find the formula *A(t)* for the amount of money in the account after *t* years.

$$\frac{dA}{dt} = \text{change in dollars per year}$$

$$\frac{dA}{dt} = \text{INTEREST ADDED PER YEAR} - \text{WITHDRAWLS PER YEAR}$$

$$\frac{dA}{dt} = 0.1A - 3600 \qquad A(0) = P_0$$

$$\int \frac{1}{0.1A - 3600}\,dA = \int dt$$

$$\frac{1}{0.1}\ln|0.1A - 3600| = t + C_1$$

$$\ln|0.1A - 3600| = \frac{1}{10}t + C_2 \qquad C_2 = \frac{1}{10}C_1$$

$$0.1A - 3600 = \pm e^{C_2} e^{\frac{1}{10}t} \qquad C_3 = \pm e^{C_2}$$

$$0.1A = C_3 e^{\frac{1}{10}t} + 3600 \qquad C = 10C_3$$

$$A = Ce^{\frac{1}{10}t} + 36{,}000$$

$$A(0) = P_0 \Rightarrow C + 36000 = P_0$$

$$\Rightarrow C = P_0 - 36000$$

$$\boxed{A(t) = (P_0 - 36000)\,e^{\frac{1}{10}t} + 36{,}000}$$

Fall 2009 Final:

The swine flu epidemic has been modeled by the Gompertz function, which is a solution of

$$\frac{dy}{dt} = 1.2\, y\, (K - ln(y)),$$

where $y(t)$ is the number of individuals (in thousands) in a large city that have been infected by time t, and K is a constant.

On July 9, 2009, 75 thousand individuals had been infected.

One month later, 190 thousand individuals had been infected.

Find the formula *y(t)* for the number of people that are infected *t* months, July 9, 2009.

$$\boxed{\lim_{t\to\infty} y(t) = e^{k} \approx 283.629}$$

$$\int \frac{1}{y(k-\ln(y))}\,dy = \int 1.2\,dt$$

$u = k - \ln(y)$

$du = -\frac{1}{y}\,dy$

$$-\int \frac{1}{u}\,du = 1.2t + c_1$$

$$-\ln|u| = 1.2t + c_1$$

$$\ln|u| = -1.2t - c_1$$

$$u = \pm e^{-c_1} e^{-1.2t}$$

$c = \pm e^{c_1}$

$$k - \ln(y) = C\,e^{-1.2t}$$

$$\ln(y) = K - Ce^{-1.2t}$$

$$y = e^{(k - Ce^{-1.2t})}$$

$$y(0) = 75 \Rightarrow k - C = \ln(75)$$

$$y(1) = 190 \Rightarrow K - Ce^{-1.2} = \ln(190)$$

$$C = K - \ln(75)$$

$$\Rightarrow k - (k - \ln(75))e^{-1.2} = \ln(190)$$

$$\Rightarrow k - ke^{-1.2} + \ln(75)e^{-1.2} = \ln(190)$$

$$\Rightarrow k(1 - e^{-1.2}) = \ln(190) - \ln(75)e^{-1.2}$$

$$\Rightarrow k = \frac{\ln(190) - \ln(75)e^{-1.2}}{1 - e^{-1.2}} \approx 5.64767$$

Side Note on Population Modeling

The Logistics Equation

Consider a population scenario where there is a limit (capacity) to the size of the population.

Let $P(t)$ = population size at time t.

M = maximum population size. (capacity)

We sometimes want a model that

a. …is like natural growth when P(t) is significantly smaller than M;
b. …levels off (with a slope approaching zero), then the population approaches M.

One such model is the so-called logistics equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) \text{ with } P(0) = P_0$$